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Robustness Bounds for Linear Systems Under Uncertainty: Eigenvalues Inside a Wedge

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This paper deals with robustness bounds for linear systems with uncertainty, where both stability and a measure of performance robustness are desired. Here, performance robustness is defined by guaranteeing that the eigenvalues of the perturbed system remain in a specific region in the complex plane such as a wedge. Using standard Lyapunov techniques, for analysis, bounds on the uncertain parameters are obtained that guarantee that the system eigenvalues remain in the desired region, whereas for synthesis, bounds and state feedback gains can also be obtained that place the closed loop system eigenvalues in the desired region. An optimization algorithm is also presented that was successfully used in the examples.

I. Introduction

MAINTAINING stability for an uncertain linear system has been studied by several authors.[1–4] Meeting additional performance related requirements, however, has received less attention. It is well known, for example, that ensuring that the system remain stable with a prescribed rate of decay (similar to having all eigenvalues to the left of a vertical line) can be handled easily by replacing the nominal system matrix A with $A + \alpha I$ in the appropriate Lyapunov equations. In Ref. 5 a variety of performance measures, such as steady-state variances of selected state variables, was used. In this paper we consider the problem of obtaining bounds on the allowable uncertainty under which the eigenvalues of the system remain in a prescribed region of the complex plane. In particular, the problem of keeping the eigenvalues inside a wedge, corresponding to a minimum damping ratio, is important in control/structure interactions (CSI).[6] Similar to Ref. 7, we extend the analysis results to address the synthesis problem. Here, we discuss obtaining bounds on the uncertain parameters, as well as the corresponding state feedback gains that guarantee that the eigenvalues of the perturbed closed-loop system will remain inside a wedge. Since we are concerned with the location of the system eigenvalues, all perturbations to the nominal system are considered time invariant but unknown.

In a recent paper[8] a similar problem is considered. There a complex Lyapunov equation is solved and its solution P is used for obtaining bounds after taking the absolute value of every entry of P. Also, the method relies on a matrix E, representing the highly structured information on the perturbation. Here, the problem of keeping all eigenvalues in a given wedge is transformed into a stability robustness problem. As such, it allows for application of most techniques developed for obtaining less conservative bounds on the uncertainty as well as obtaining state feedback gains for the synthesis problem. We also discuss the connections between these results and those of Ref. 9 and the effects of these results on optimizing the allowable uncertainty. Examples are provided, including an analysis of an active damping system for a 10-mode model of a dual keel space station. In the Appendix an optimization algorithm is presented that was successfully used in the examples.

II. Eigenvalues Inside a Wedge

Consider the nominal system

$$\dot{x} = Ax, \qquad x \in \mathbf{R}^n \tag{1}$$

with all eigenvalues of A in a specific wedge, as in Fig. 1. This implies that all eigenvalues of A have damping ratios equal to or greater than $\sin\theta$. Clearly, the eigenvalues of $\hat{A} = e^{i\theta}A$ are to the left of the imaginary axis, and $e^{i\theta}A$ is a stable matrix. The following system, therefore, is stable:

$$\dot{z} = \hat{A}z = e^{i\theta}Az, \qquad z \in \mathbf{C}^n \tag{2}$$

In addition, the Lyapunov equation

$$Pe^{i\theta}A + e^{-i\theta}A^TP = -Q = -Q_1 - iQ_2 \tag{3}$$

has, for any hermitian positive definite Q, a unique hermitian positive definite solution $P = P^*$,[10] where the asterisk denotes complex conjugate transpose. Clearly,

$$P = P_1 + iP_2, \qquad P_1 > 0, \qquad P_2 = -P_2^T \tag{4}$$

Similarly, the matrix $e^{-i\theta}A$ is also stable, and, for any hermitian positive definite $\hat{Q}$, the Lyapunov equation

$$Re^{-i\theta}A + e^{i\theta}A^TR = \hat{Q} \tag{5}$$

has a unique hermitian positive solution R. By examining Eq. (3) or by direct substitution, it is clear that if $\hat{Q} = \bar{Q}$, where $\bar{Q}$ is the complex conjugate of Q, we have $R = \bar{P} = P_1 - iP_2$. Separating real and imaginary parts of Eqs. (3) and (5) results in four equations that can be put into the following matrix form:

$$\tilde{P}\tilde{A} + \tilde{A}^T\tilde{P} = -\tilde{Q} \tag{6}$$

where

$$\tilde{A} = \begin{bmatrix} A\cos\theta & -A\sin\theta \\ A\sin\theta & A\cos\theta \end{bmatrix}, \qquad \tilde{P} = \begin{bmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{bmatrix}$$
$$\tilde{Q} = \begin{bmatrix} Q_1 & -Q_2 \\ Q_2 & Q_1 \end{bmatrix} \tag{7}$$

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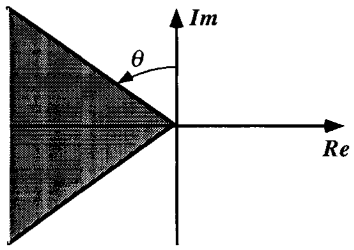

Fig. 1 Desired region for preserving a minimum damping ratio.

Equation (6) simply transforms the two $n \times n$ complex matrix Eqs. (3) and (5) into an equivalent $2n \times 2n$ real matrix equation. Indeed, since positive definiteness of P and Q in Eq. (3) implies positive definiteness of $\tilde{P}$ and $\tilde{Q}$ in Eq. (6), this development can be seen as an alternative for the result of Ref. 9 (i.e., eigenvalues of A are inside a wedge if and only if eigenvalues of $\tilde{A}$ are stable). By using Eq. (6), standard techniques can be used for obtaining not only P in Eq. (3) but also uncertainty bounds for stability robustness, as discussed subsequently.

Consider the system in Eq. (2) with parameter uncertainty of the form

$$\dot{x} = (A + \Delta A)x, \qquad x \in \boldsymbol{R}^n \tag{8}$$

where ΔA contains the uncertain parameters. Since we focus on eigenvalue assignment, we require the uncertainty to be constant but unknown. It is desired to find bounds on ΔA such that the eigenvalues of the system in Eq. (8) remain inside the wedge of Fig. 1. A necessary and sufficient condition for this is to have all of the eigenvalues of $e^{i\theta}(A + \Delta A)$ and $e^{-i\theta}(A + \Delta A)$ strictly to the left of the imaginary axis. As discussed subsequently, since A and ΔA are real matrices, we need to consider $e^{i\theta}(A + \Delta A)$ only. For this, let

$$\dot{z} = e^{i\theta}(A + \Delta A)z, \qquad z = z_1 + iz_2 \in \boldsymbol{C}^n, \qquad z_i \in \boldsymbol{R}^n \tag{9}$$

For the system in Eq. (9), we choose $V(z) = z^*Pz$ as a Lyapunov function, and for the derivative of V we have

$$\dot{V}(z) = z^*[e^{i\theta}P(A + \Delta A) + e^{-i\theta}(A + \Delta A)^TP]z \tag{10}$$

Choosing P as the solution of Eq. (3) [or obtained from the solution of Eq. (6)] yields

$$\dot{V}(z) = z^*[e^{i\theta}P\,\Delta A + e^{-i\theta}\Delta A^TP - Q]z \tag{11}$$

Note that for $\dot{z} = e^{-i\theta}(A + \Delta A)z$, one can solve Eq. (5) for R and use $V_2(z) = z^*Rz$ as the Lyapunov function.

It is easy to show that if $\dot{V}$ in Eqs. (10) and (11) is negative, so is the derivative of V_2. After some standard manipulations, Eqs. (10) and (11) can be written as

$$\dot{V}(z) = [z_1^T \; z_2^T][\tilde{P}(\tilde{A} + \Delta\tilde{A}) + (\tilde{A} + \Delta\tilde{A})^T\tilde{P}]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{12}$$

and

$$\dot{V}(z) = [z_1^T \; z_2^T][\tilde{P}\,\Delta\tilde{A} + \Delta\tilde{A}^T\tilde{P} - \tilde{Q}]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{13}$$

respectively, where

$$\Delta\tilde{A} = \begin{bmatrix} \Delta A \cos\theta & -\Delta A \sin\theta \\ \Delta A \sin\theta & \Delta A \cos\theta \end{bmatrix}$$

It is clear that the derivative of the Lyapunov function is negative if

$$[\tilde{P}(\tilde{A} + \Delta\tilde{A}) + (\tilde{A} + \Delta\tilde{A})^T\tilde{P}] = [\tilde{P}\,\Delta\tilde{A} + \Delta\tilde{A}^T\tilde{P} - \tilde{Q}] < 0 \tag{14}$$

The following sufficient condition can now be stated:

Lemma 1: Let eigenvalues of matrix A be inside a given wedge in the complex plane (Fig. 1). Then, the eigenvalues of the system in Eq. (8) remain inside the wedge if the condition of Eq. (14) is satisfied.

Bounds on the uncertainty matrix ΔA of the system in Eq. (8) can now be obtained by using several available techniques (that have been developed for stability robustness). For example, let $\Delta A = \Delta A(r)$ be a linear function of the parameter vector $r = [r_1, \ldots, r_m]^T \in \boldsymbol{R}^m$, where r is unknown but constant. The derivative of the Lyapunov function $\dot{V}$ is multilinear in the parameter vector r and, thus, the maximum value of $\dot{V}$ is attained at one or more of the vertices of the hypercube $|r_i| \leq k$.[11-12] Defining $\Delta\tilde{A}^{(j)} = \Delta\tilde{A}(r^{(j)})$ where $r^{(j)}$ is a vertex of the hypercube $|r_i| \leq k$, we can state the following sufficient condition.

Lemma 2: If ΔA depends linearly on the parameter vector r, then the eigenvalues of the system in Eq. (8) remain inside the wedge of Fig. 1 for the hypercube $|r_i| \leq k$, if there exists a $\tilde{P} > 0$ such that Eq. (14) is satisfied for each of the 2^m matrices $\Delta\tilde{A}^{(j)}$.

Among the techniques for computing bounds directly is Ref. 1. Following Ref. 1, let $\tilde{Q} = LL^T$, where L is a real $2n \times 2n$ lower triangular matrix with positive diagonal elements. Also define

$$W(r) = L^{-1}[\tilde{P}\,\Delta\tilde{A}(r) + \Delta\tilde{A}(r)^T\tilde{P}]L^{-T} \tag{15}$$

and

$$k = \frac{1}{\max_r \{\bar{\sigma}[W(r)]: \; |r_i| \leq 1\}} \tag{16}$$

where $\bar{\sigma}[\cdot]$ is the maximum singular value of $[\cdot]$. Note that, due to linear dependency on r and convexity of the maximum singular value, k can be obtained by checking only half of the vertices of the hypercube $|r_i| \leq 1$. As a consequence of the results of Ref. 1, we have the following sufficient condition.

Lemma 3: If ΔA depends linearly on the parameter vector r, then the eigenvalues of the system in Eq. (8) remain inside the wedge of Fig. 1, if the vector r is inside the hypercube $|r_i| \leq k$, where k is defined by Eq. (16).

Other stability robustness techniques, such as those used by Yedavalli[3,13] or Khargonekar and Zhou,[2] can also be used in conjunction with these results to study the eigenvalue placement inside a wedge. In particular, optimizing over elements of Q, as in Ref. 1, or state transformation, as in Ref. 14, can be used to obtain less conservative bounds. In the next section, we will study this issue in more detail. Finally, note that if $\theta = 0$, the $2n \times 2n$ problem separates into two identical $n \times n$ stability robustness problems.

Remark: To obtain bounds for allowable uncertainty when the region of interest is not a wedge and has a more general shape, we can use the following approach. Let the line L intersect the real axis at $(-\alpha + i0)$ with an angle θ from the imaginary axis (counter-clockwise). Then $e^{-i\theta}(A + \alpha I)$ is stable if eigenvalues of A are to the left of L (or $-e^{-i\theta}(A + \alpha I)$ if they are to the right of L). Let the region of interest be approximated by a convex region enclosed by a finite number of line segments. The extension of each line segment L^j divides the complex plane into two half planes and intersects the real axis at the point $(-\alpha + i0)$, making an angle (counter-clockwise) of θj with the imaginary axis. For each j, a bound on ΔA (say k_j) can be calculated through the method of Sec. II. Naturally, the bound for the region is the smallest value of k_j. Note that, for any two lines symmetric with respect to the real axis, only one k_j needs to be calculated.

III. Reducing Conservatism

A natural approach for obtaining less conservative bounds is to use optimization methods. By using the sufficient condition of Lemma 2, the problem of determining whether a given size uncertainty is quadratically stable can be formulated as a (nondifferentiable) convex programming problem. Foolproof methods for minimizing convex functions can be found in Ref. 15, and, in particular, in Ref. 12. This allows one to determine whether the system in Eq. (8) or Eq. (9) is quadratically stable for a given size of uncertainty.

Evaluating the quadratic stability of the system in Eq. (9) by imposing the structure of Eq. (7) poses an interesting question. By disregarding the specific structure of Eq. (7) on $\tilde{P}$ or, equivalently, $\tilde{Q}$, can less conservative bounds be obtained? Note that such an approach would involve $2n^2 + n$ parameters in the optimization, as opposed to n^2 parameters when the structure of Eq. (7) is imposed. Consider Eq. (6), a real $2n \times 2n$ Lyapunov equation, with no structure imposed on $\tilde{P}$ (i.e., it is only required to be positive definite). Suppose that the optimization over $\tilde{P}$ is used to determine that a given uncertainty is quadratically stable, that is,

$$\tilde{P}(\tilde{A} + \Delta\tilde{A}) + (\tilde{A} + \Delta\tilde{A})^T\tilde{P} < 0$$

Without loss of generality, let

$$\tilde{Q} = \begin{bmatrix} q_1 & q_2^T \\ q_2 & q_3 \end{bmatrix}, \qquad \tilde{P} = \begin{bmatrix} p_1 & p_2^T \\ p_2 & p_3 \end{bmatrix} \tag{17}$$

where p_i and q_i are $n \times n$ real matrices. Let

$$\tilde{P} = \tilde{P}_1 + \tilde{P}_2, \qquad \tilde{P}_1 = \frac{1}{2}\begin{bmatrix} p_1 + p_3 & p_2^T - p_2 \\ p_2 - p_2^T & p_3 + p_1 \end{bmatrix}$$

$$P_2 = \frac{1}{2}\begin{bmatrix} p_1 - p_3 & p_2^T + p_2 \\ p_2 + p_2^T & p_3 - p_1 \end{bmatrix}$$

Naturally,

$$M + H = \tilde{P}(\tilde{A} + \Delta\tilde{A}) + (\tilde{A} + \Delta\tilde{A})^T\tilde{P} < 0 \tag{18}$$

where

$$M = \tilde{P}_1(\tilde{A} + \Delta\tilde{A}) + (\tilde{A} + \Delta\tilde{A})^T P_1$$

and

$$H = \tilde{P}_2(\tilde{A} + \Delta\tilde{A}) + (\tilde{A} + \Delta\tilde{A})^T\tilde{P}_2$$

The structure of M and H can be exploited further. Consider $2n$ vectors $y_a^T = [y_1^T y_2^T]$ and $y_b^T = [-y_2^T y_1^T]$ for some y_1 and y_2. It is simple to show

$$y_a^T M y_a = y_b^T M y_b \qquad \text{and} \qquad y_a^T H y_a = -y_b^T H y_b \tag{19}$$

Next, from Eq. (18), for any y_a

$$y_a^T M y_a + y_a^T H y_a < 0 \tag{20}$$

and for any y_b

$$y_b^T M y_b + y_b^T H y_b < 0 \tag{21}$$

Substituting Eq. (19) into Eq. (21) yields

$$y_a^T M y_a - y_a^T H y_a < 0 \tag{22}$$

Comparing Eqs. (22) and (20), we see that for y_a

$$y_a^T M y_a < 0 \tag{23}$$

Since it is straightforward to show that, if $\tilde{P}$ is positive definite, so is $\tilde{P}_1$, the Lyapunov function could have been constructed with $\tilde{P}_1$. Further, $\tilde{P}_1$ is the unique positive definite solution to Eq. (6) with

$$\tilde{Q} = \frac{1}{2}\begin{bmatrix} q_1 + q_3 & q_2^T + q_2 \\ q_2 - q_2^T & q_3 + q_1 \end{bmatrix}$$

where q_i is from Eq. (17). In summary, the same result could have been obtained by imposing the structure of Eq. (7) on $\tilde{P}$, which has considerably fewer parameters over which the optimization is performed.

As mentioned previously, techniques for minimizing our convex function are given in Ref. 12. We have modified one of these techniques to include an iteration on k. The resulting algorithm, given in the Appendix, has been successfully applied to the following examples.

IV. Examples

A. Example 1

Consider Example 1 of Ref. 8, where

$$A = \begin{bmatrix} -2 & -1 \\ 3 & -6 \end{bmatrix}$$

As in Ref. 8, it is desired to obtain bounds on the uncertainty so that the eigenvalues remain inside the wedge $\theta = 45$ deg. In Ref. 8 the bound obtained was

$$|\Delta a_{ij}| < 0.9053 e_{ij} \qquad \text{for} \qquad E = \begin{bmatrix} 0.6 & 0.2 \\ 1.0 & 2.0 \end{bmatrix}$$

where E represents structural information about the uncertainty, although it is not clear how this matrix was obtained. Nonetheless, to accommodate this information, we can assume the following structure for ΔA:

$$\Delta A = \begin{bmatrix} 0.6r_1 & 0.2r_2 \\ r_3 & 2r_4 \end{bmatrix}$$

Following the development of Lemma 3, we obtain the following bound for $Q = I$:

$$|r_i| < 1.2927$$

It is important to note that in addition to the improvement shown previously (i.e., increasing the bound from 0.9053 to 1.2927 for the same Q), the framework here allows the use of almost all of the methods obtained for stability robustness, such as those in Ref. 5. In particular, we may use optimization techniques, such as the algorithm outlined in the Appendix. This yields a bound of

$$|r_i| < 1.4172$$

Thus, optimization yielded an improvement of 20% in the bound.

B. Example 2

Consider the example used in Ref. 2 (also used in Ref. 1 as Example 1). In this example, matrix A is the closed-loop system matrix given by

$$A = A_{ol} + BKC$$

where the open-loop system matrix is

$$A_{ol} = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 + r_1 & -0.7070 & 1.4200 + r_2 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 + r_3 & -7.5922 \\ -5.52 & 4.49 \\ 0.0 & 0.0 \end{bmatrix}, \qquad K = \begin{bmatrix} -0.99633980 \\ 1.801833665 \end{bmatrix}$$

with $C = [0\ 1\ 0\ 0]$. The matrix ΔA has the form

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.99633989r_3 & 0 & 0 \\ 0 & r_1 & 0 & r_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix K is taken from Ref. 2, where it was chosen by an optimization technique. The resulting nominal (i.e., $r_i = 0$) closed-loop system matrix has two real eigenvalues $(-0.736, -18.3963)$ and a pair of complex conjugate eigenvalues $(-0.2676 \pm i1.2501)$. The nominal eigenvalues are thus inside a wedge with $\theta = 11.2$ deg (or 19.4% critical damping).

Results for two damping requirements, using the method of Lemma 3 with $Q = I$ and the algorithm given in the Appendix, are given in Table 1. As illustrated in these results, optimization is clearly important for determining less conservative bounds when using Lyapunov techniques.

C. Example 3

As a more practical example, consider a NASTRAN model of a dual keel space station, as shown in Fig. 2. A vibration control system, using collocated consistent proof-mass actuators and linear rate sensors, is to be used to control the vibration of the upper and lower booms by adding damping to a subset of the most important modes of the model. A modal cost function equal to the root-sum-square of the magnitude of the rotation of three points along the upper and lower boom (see Fig. 2) per disturbance force at node 1 was evaluated for each mode. The 10 bending modes with the largest cost were selected. Five collocated sensor/actuator pairs are to be used to add damping to these modes (see Fig. 2).

The closed-loop system matrix is given by

$$A = A_{ol} + BKB^T$$

where K is the controller gain matrix, B a 20×5 matrix containing modal deflection data, and A_{ol} a 20×20 matrix containing the open-loop modal frequencies and damping ratios of the structure. The matrices A_{ol} and B are of the form

$$A_{ol} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{10} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{10} \end{bmatrix}$$

where

$$A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix} \quad \text{and} \quad B_i = \begin{bmatrix} 0 & 0 \cdots & 0 \\ \phi_{i1} & \phi_{i2} \cdots & \phi_{ip} \end{bmatrix}$$

and where ω_i is the modal frequency of the ith mode, ζ the open loop damping ratio, and ϕ_{ij} the modal deflection of the ith mode at the jth sensor/actuator location. The data for this example are contained in Tables 2–4. The open-loop modal frequencies and damping ratios are given in Table 2, and the modal deflections that make up the B matrix are contained in Table 3. Here the second row of the matrix B_i is the ith row of Table 3. For this example, the controller gain matrix is diagonal and is given in Table 4 (see Ref. 11 for details). The resulting closed-loop frequencies and damping ratios are given in Table 2.

The robust performance of this system is to be evaluated by determining the bounds on open-loop modal frequencies that

Table 1 Optimization yields significantly better robustness bounds

Wedge angle, deg	Damping requirement, %	Lemma 3 $Q = I$	Lemma 2 optimization	Factor improvement
10.00	17.36	0.0224	0.1400	6.25
2.87	5.00	0.0884	0.4300	4.86

Table 2 Open- and closed-loop frequencies and damping ratios

Design mode	Flex mode	Open loop ω, r/s	Open loop ζ, %	Closed loop ω, r/s	Closed loop ζ, %
1	1	0.569	0.5	0.569	5.75
2	2	0.589	0.5	0.606	22.04
3	12	0.873	0.5	0.888	7.11
4	13	1.004	0.5	0.995	7.26
5	14	1.176	0.5	1.188	16.67
6	17	1.433	0.5	1.448	6.24
7	18	1.549	0.5	1.544	6.17
8	19	1.840	0.5	1.817	5.01
9	20	1.939	0.5	1.899	11.08
10	21	2.227	0.5	2.178	15.30

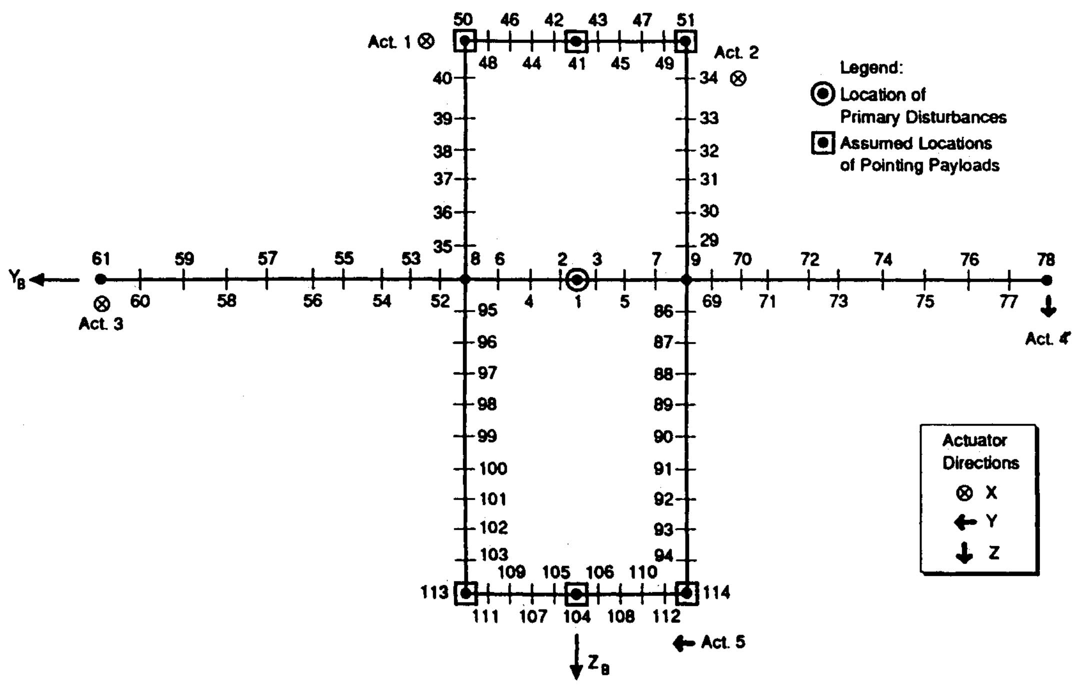

Fig. 2 NASTRAN model of a dual keel space station showing assumed payload and disturbance locations.

Table 3 Modal deflection data, B matrix

Flex mode	Sensor/actuator number				
	1	2	3	4	5
1	−0.00273	−0.00202	0.01148	0.09129	0.00189
2	0.01107	0.01284	−0.08283	0.01046	−0.00004
12	−0.03385	−0.05651	0.01333	0.01052	−0.00288
13	0.01937	0.04371	0.04167	0.00268	−0.00091
14	0.06005	−0.06815	0.05617	−0.00177	0.01502
17	−0.01973	0.03873	−0.01947	−0.00927	0.02528
18	−0.02351	−0.06864	−0.01816	−0.00373	0.00615
19	−0.02165	0.07503	0.01461	−0.00429	−0.01200
20	0.00000	−0.01552	−0.00509	−0.03095	−0.04706
21	0.15074	−0.03003	−0.01422	−0.00195	−0.00130

Table 4 Controller gains

S/A number	Gain, lb-s/in.
1	27.0
2	28.0
3	34.0
4	7.0
5	170.0

guarantee the closed-loop system will maintain at least 4% of critical damping in each mode in the design model. To model uncertainty in the modal frequency of a bending mode, we note that the magnitude of the eigenvalue of the 2×2 matrix $(1 + r_i)A_i$ is $(1 + r_i)\omega_i$. This may be shown by examining the perturbed modal equation for the ith mode:

$$\ddot{z} = -2\zeta\omega_i(1 + r_i)\,\dot{z}_i - \omega_i^2(1 + r_i)^2 z_i + \phi_i u$$

Letting

$$w_i = \begin{bmatrix} z_i \ (1 + r_i) \\ \dot{z}_i \end{bmatrix}$$

the perturbed modal equation for one mode in state-space form is given by

$$\dot{w}_i = (A_i + r_i A_i)\, w_i + B_i u$$

Here r_i is the constant but unknown parameter, which represents the percent variation in the ith modal frequency. Because we are only interested in positive modal frequencies, we require that $r_i > -1$ and also note that the nominal value of r_i is zero. To represent an uncertainty on the modal frequency for each bending mode in the model, the perturbed system in state-space form is given by

$$\dot{x} = [A_{\text{ol}} + B\,K\,B^T + \Delta A(r)]x$$

where

$$\Delta A = A_{\text{ol}}\Delta(r)$$

and where $\Delta(r) = \text{diag}[r_1, r_1, r_2, r_2, \ldots, r_{10}, r_{10}]$. Note that even though modal frequency appears in a nonlinear fashion in the system matrix, modal frequency uncertainty may be represented by a linear model.

The bound on the modal frequencies that guarantees 4% damping was found to be 1.9%. This bound assumes all 10 modal frequencies may vary independently. This assumption, however, is conservative since some modal frequencies may vary together. For example, the first two flex modes of the structure are the x-axis and z-axis bending modes of the transverse boom. This implies that the uncertainty of the modal frequencies of these modes is highly correlated. Thus, a more realistic uncertainty model is one that varies certain modal frequencies together. Therefore, noting that the selected modes contain three groups of consecutive modes, bounds were also computed for a more structured variation, specifically with $\Delta(r) = \text{diag}[r_1 I_4, r_2 I_6, r_3 I_{10}]$. This varies modes 1 and 2 together, modes 12–14 together, and modes 17–21 together. The bound that guarantees 4% damping for this case was found to be 9.1%. This is a reasonable bound within which the modal frequencies may actually be estimated.

V. Discussion

The results presented thus far allow one to analyze the robust performance of a system. These results may be easily extended, along the lines of results for stability of Ref. 7, to synthesis for systems using state feedback. Consider the uncertain plant

$$\dot{x} = [A + \Delta A(r)]x + [B + \Delta A(r)]u$$
$$x \in R^n, \qquad u \in R^p, \qquad r \in R^m \tag{24}$$

where $\Delta A(r)$ and $\Delta B(r)$ are linear functions of the parameter vector r. Define $\Delta A^{(j)} = \Delta A(r^{(j)})$ and $\Delta B^{(j)} = \Delta B(r^{(j)})$ where $r^{(j)}$ is a vertex of the hypercube $|r_i| \le k$ and

$$M_j = [(\text{A} + \Delta A^{(j)})P_{11} + (B + \Delta B^{(j)})P_{21}]$$
$$+ [(\text{A} + \Delta A^{(j)})P_{11} + (B + \Delta B^{(j)})P_{21}]^T \tag{25}$$

Recall from Ref. 7 that the system in Eq. (24) can be robustly stabilized if there exists a real symmetric matrix $P_{11} > 0$ and a real matrix P_{21} such that the $2m$ matrices M_j are negative definite (see Ref. 7 or Ref. 16 for proof). The stabilizing control law is given by $u = P_{21}P_{11}^{-1}x$, and the corresponding Lyapunov function is $V(x) = x^T P_{11}^{-1} x$. This synthesis result may also be applied to the problem of keeping eigenvalues inside a wedge. Here, we simply transform to the $2n$ system, where $\tilde{A}$ is used instead of A and $\Delta\tilde{A}$ instead of ΔA, etc. However, to obtain a control law that may be implemented, we must have

$$\tilde{K} = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} = \begin{bmatrix} P_{211} & P_{212} \\ P_{212} & P_{211} \end{bmatrix} \begin{bmatrix} P_{111} & -P_{112} \\ P_{112} & P_{111} \end{bmatrix}^{-1} = \tilde{P}_{21}\tilde{P}_{11}^{-1} \tag{26}$$

which results in a nonlinear constrained optimization problem. Obviously, one way to deal with this problem is to optimize over P_{111} and P_{211} with $P_{112} = 0$ and $P_{212} = 0$. Removing the nonlinear constraint allows the optimization algorithm developed for analysis to be used for synthesis with only slight modifications, as discussed in the Appendix.

This method can also be applied directly to the complex ($n \times n$) formulation. In that case it can be shown, similar to the development of Sec. II, that if there exist a hermitian matrix $P_{11} > 0$ and a complex matrix P_{21} such that

$$M_j = e^{i\theta}[(A + \Delta A^{(j)})P_{11} + (B + \Delta B^{(j)})P_{21}]$$
$$+ e^{-i\theta}[(A + \Delta A^{(j)})P_{11} + (B + \Delta B^{(j)})P_{21}]^*$$

is negative definite for all vertices of the hypercube, then the control law

$$K = P_{21}P_{11}^{-1}$$

results in the closed-loop eigenvalues inside the wedge of Fig. 1 for all parameter values inside the hypercube. To obtain a realizable control law, K must be real. The resulting nonlinear constrained optimization may be avoided by optimizing over real P_{11} and real P_{21}, which is equivalent to setting P_{112} and P_{212} in Eq. (26) to zero, as discussed previously.

VI. Conclusions

We investigated the use of quadratic Lyapunov functions for analyzing the robust performance of an uncertain system. The measure of performance was maintaining a given amount of damping in the perturbed closed-loop system. Sufficient conditions for achieving robust performance were presented, and the analysis procedure was given in an optimization setting. It was also shown that the sufficient conditions could be formulated using either a complex system of order n or a real system of order $2n$, both of which will yield equivalent results. An unconstrained optimization algorithm was presented and successfully used on example problems. The formulation allows for the synthesis of state feedback gains that correspond to the achieved level of robust performance. In that case, obtaining a realizable control law results in a nonlinear constrained optimization. This constraint may be alleviated with the appropriate choice of an optimization basis, in which case the optimization algorithm outlined in the Appendix is also applicable.

Appendix

The analysis of the robust performance of a system via Lyapunov techniques requires optimization of the Lyapunov function to obtain the least conservative results. For the problem of keeping eigenvalues inside a wedge, the analysis may be performed by analyzing the quadratic stability of a complex system of order n or a real system of order $2n$, both of which have been shown to be equivalent. First, consider the complex formulation. The problem of determining whether a given size uncertainty is quadratically stable can be formulated as a convex programming problem. That is, for a given k,

$$\min_P \left\{ \max_{j=1:2^m} \{\lambda_{\max}[M_j(P)]\} : P = P^* > 0 \right\}$$

is a convex programming problem. For analysis, we use

$$M_j(P) = e^{i\theta} P(A + \Delta A^{(j)}) + e^{-i\theta}(A + \Delta A^{(j)})^T P$$

whereas for synthesis, we use

$$M_j(P) = e^{i\theta}[(A + \Delta A^{(j)})P_{11} + (B + \Delta B^{(j)})P_{21}]$$
$$+ e^{-i\theta}[(A + \Delta A^{(j)})P_{11} + (B + \Delta B^{(j)})P_{21}]^*$$

A foolproof method for minimizing convex functions is the subgradient method.[15]

Following Ref. 12, let $P_1 \cdots P_l$ be a basis for the subspace $\{P \mid P = P^* \in \boldsymbol{P}\}$. Here $\boldsymbol{P}$ is the space of matrices that the optimization is to be performed over. For example, since we are doing analysis with the complex formulation, then $\boldsymbol{P}$ is the set of all complex $n \times n$ hermitian matrices. Thus, for $n = 2$, a basis of all complex hermitian 2×2 matrices is

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad P_4 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

We could avoid using complex matrices by formulating the problem in terms of real matrices with the structure of Eq. (7) imposed on P. For this case a basis is

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad P_4 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

The system in Eq. (8) is stable if there exists $a_1 \cdots a_l \in \boldsymbol{R}$ such that

$$P = \sum_{i=1}^{l} a_i P_i > 0 \quad \text{and} \quad M_j(P) < 0; \qquad j = 1, 2^m$$

To determine this, define the functions $h_0, h_1, \cdots$, as

$$h_0(a) = \lambda_{\max}[-P] \quad \text{and} \quad h_j(a) = \lambda_{\max}[M_j(P)]; \qquad j = 1, 2^m$$

and define $f(a)$ as

$$f(a) = \max_{j=0:2^m} h_j(a)$$

If $h_0(a) = f(a)$, then let u be a unit length eigenvector of $-P$ associated with its maximum eigenvalue and set $\delta a_i = -u^* P_i u$; $i = 1, l$. If $h_j(a) = f(a)$, then let u be a unit length eigenvector of $M_j(P)$ associated with its maximum eigenvalue and set $\delta a_i = u^* [M_j(P_i)] u$; $i = 1, l$.

The vector $-\delta a$ is the search direction, and thus the subgradient method would generate a new a equal to $a - (\alpha / s \|\delta a\|)\delta a$. This generates the sequence

$$a^{(s+1)} = a^{(s)} - \frac{\alpha}{s\|\delta a\|}\delta a$$

which is guaranteed to converge to the minimizer of $f(a)$. This yields the $P = P^* > 0$ which minimizes $\max_{j=1:2^m}\{\lambda_{\max}[M_j(P)]\}$ for a given k.

An algorithm that combines an iteration on k with this technique outlined was used to determine the bounds presented in the paper. The algorithm is motivated from the observation that the optimum a for some k will be close to the optimum a for some $k + \epsilon$. Also, since the objective is to determine if the system in Eq. (9) is quadratically stable for a given k, it is not necessary to minimize $f(a)$, but simply to determine if there exists an a such that $f(a) < 0$.

Algorithm:

1) Initialization.
   a) Solve Eq. (3) for P with $Q = I$.
   b) Determine a from P and the defined basis.
   c) Calculate the bound k, using Eqs. (15) and (16), with $L = I$.
   d) Set $s = s_0$, some initial iteration number.
   e) Set Δk, the incremental uncertainty size.
   f) Set $s_{\max}$, the maximum number of iterations.
   g) Set α, the constant step size.
2) Perform the following computations.
   a) $s = s + 1$
   b) Compute $f(a)$ and δa as outlined previously.
   c) $a = a - (\alpha / s\|\delta a\|)\delta a$
   d) $a = (1/\|a\|)a$
3) If $f(a) < 0$, then do the following; otherwise, go to step 4.
   a) $k_{\max} = k$
   b) $k = k + \Delta k$
   c) Go to step 2.
4) If $s - s_0 < s_{\max}$, then go to step 2); otherwise, go to step 5.
5) Stop.

The algorithm essentially increases k until it can no longer find an a that yields $f(a) < 0$ and reaches the maximum iteration number. The values of the parameters α, s_0, and Δk will determine how fast the algorithm will approach the largest value of k. For most of the problems considered in this paper, Δk was set on the order of one-tenth to one-hundredth of the value of k determined in step 1(c), s_0 was set to 1000, α was set equal to 1, and $s_{\max}$ was set to 5000. This allows small steps of approximately the same size to be taken as the algorithm proceeds.

To illustrate how the algorithm proceeds, Fig. A1 shows the bound k plotted vs the number of iterations ($s - s_0$) for case 2 of Example 3. Note that the algorithm initially increases k rapidly and that between iteration 2000 and 5000 there were

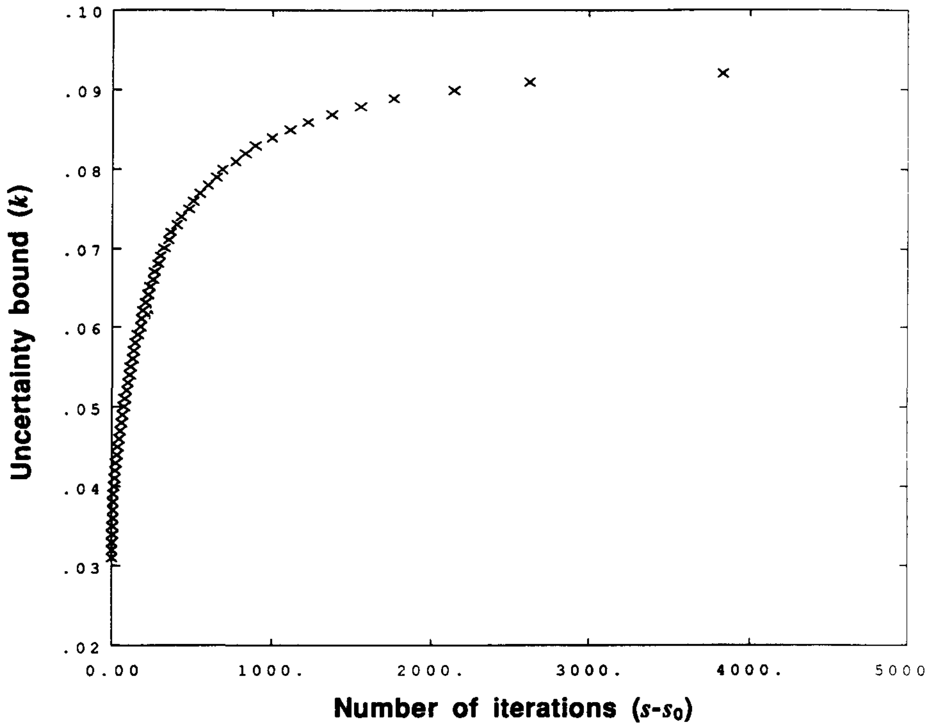

Fig. A1 Achieved uncertainty bound vs number of iterations.

only three increases in k. Thus, by examining this plot we may conclude that for all practical purposes the algorithm has converged.

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